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Recovering divisor classes via their (T)-adic filtrations

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Abstract

Continuing the investigation of the canonical map of divisor class groups $j^*: \mathrm{Cl}(B) \to \mathrm{Cl}(B/tB)$, we consider what happens if $\ker j^*$ is nonzero. The first result gives an injection of $\mathrm{Cl}(B)$ into $\prod_{n\geq 1} \mathrm{Cl}(B/t^nB)$, where the definition of class group is extended to include the rings B/t^nB. We also get an action of $\ker j^*$ on small MCM B-modules, resulting, in the case of a ring A without DCG and with a small MCM module C, in infinitely many distinct small MCM $A[[T]]$-modules lifting C. © 1998 Elsevier Science B.V. All rights reserved.

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0. Introduction

In commutative algebra it is often interesting to look at "Lefschetz-type" questions, i.e. how properties of a commutative noetherian ring S transfer to properties of hypersurface cuts S/fS of S, and vice versa.

In this paper we consider two such problems. First, we compare the divisor class group of a commutative Noetherian ring B to the divisor class group of a hypersurface section B/fB of B. Second, we use elements of the kernel of a certain map between the divisor class groups of B and B/fB to construct new finitely generated maximal Cohen–Macaulay modules from given modules of this kind.

The divisor class group $\mathrm{Cl}(B)$ of a normal domain B measures how close B is to having unique factorization, in the sense that it is trivial if and only if B has unique factorization.

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Samuel asked, for a complete unique factorization domain A, whether the formal power series ring $A[[T]]$ has unique factorization [17]. To study Samuel's question, Danilov looked at the more general problem, for any normal domain A, of comparing the divisor class groups $\mathrm{Cl}(A)$ and $\mathrm{Cl}(A[[T]])$ [3–6].

Danilov considered the homomorphism

$$i^* : \mathrm{Cl}(A) \to \mathrm{Cl}(A[[T]])$$

and its left splitting

$$j^* : \mathrm{Cl}(A[[T]]) \to \mathrm{Cl}(A).$$

The ring A is said to "have discrete divisor class group" if i^* is bijective. Since j^* is a left splitting for i^*, this is equivalent to j^* being injective. Unlike unique factorization, discreteness of the divisor class group is a local property of the ring and is reflected by faithfully flat extensions, and hence easier to study.

Using techniques of algebraic geometry, Danilov explored conditions for a ring to have discrete divisor class group. He answered Samuel's question in the affirmative under the additional assumption that the residue field of A is algebraically closed. He also characterised excellent normal $\mathbb{Q}$-algebras with discrete divisor class group, thus showing that many such rings have non-discrete divisor class group.

We consider the more general setting introduced by Lipman: let B and B/tB be normal domains, and let j^* be the canonical homomorphism from the divisor class group of B to that of B/tB (see [13]). To examine j^* more closely, we consider class groups of the rings B/t^nB for $n \geq 1$. As B/t^nB is not a normal domain if $n > 1$, we first extend the definition of divisor class group to cover these rings.

We let j_1^* be j^*, and we define, for $n \geq 2$, homomorphisms $j_n^* : \mathrm{Cl}(B) \to \mathrm{Cl}(B/t^nB)$ and $\psi_n : \mathrm{Cl}(B/t^nB) \to \mathrm{Cl}(B/t^{n-1}B)$. These maps are compatible with each other and so provide a map

$$\tilde{j} : \mathrm{Cl}(B) \to \varprojlim \mathrm{Cl}(B/t^nB).$$

The main result is the following theorem.

Theorem 5.3. *If B is t-adically complete, then the map $\tilde{j}$ is injective.*

Maximal Cohen–Macaulay modules are those modules with depth equal to the dimension of the ring. It is still unknown whether finitely generated maximal Cohen–Macaulay modules exist over every commutative Noetherian ring. We prove the following theorem in order to create, for certain rings B, new finitely generated maximal Cohen–Macaulay B-modules from old ones. Let $\mathfrak{M}$ be the set of isomorphism classes of finitely generated maximal Cohen–Macaulay B-modules.

Theorem 6.1. *Suppose B and B/tB are normal domains such that B/tB satisfies R_2. Then the kernel of the map $j^* : \mathrm{Cl}(B) \to \mathrm{Cl}(B/tB)$ acts on $\mathfrak{M}$ in the following way: if $[D] \in \mathfrak{M}$ and $[\mathfrak{a}] \in \ker j^*$, then*

$$[\mathfrak{a}] \cdot [D] = [\mathrm{Hom}_B(\mathfrak{a}, D)].$$

This action has the following properties:

(a) *If* $[D], [E] \in \mathfrak{M}$ *are in the same orbit, then* $D/tD \cong E/tE$.

(b) *Let* $[D] \in \mathfrak{M}$ *and* $n = \operatorname{rank}_B(D)$. *If* $[\mathfrak{a}], [\mathfrak{b}] \in \ker j^*$ *are such that* $n[\mathfrak{a}] \neq n[\mathfrak{b}]$, *then* $[\mathfrak{a}] \cdot [D] \neq [\mathfrak{b}] \cdot [D]$.

Statement (b) of Theorem 6.1 implies that this action is free when $\ker j^*$ is torsion-free. Thus, when $\ker j^*$ is torsion-free and non-zero, this action produces from a single finitely generated maximal Cohen–Macaulay B-module infinitely many such modules. Moreover, by statement (a), all these modules are lifts of the same B/tB-module.

Hochster gives an example of a finitely generated maximal Cohen–Macaulay module C over a certain $\mathbb{Q}$-algebra A (see [10]). By Danilov's characterisation, A has non-discrete divisor class group, and, by a theorem of Griffith and Weston, the kernel of $j^* : \mathrm{Cl}(A[[T]]) \to \mathrm{Cl}(A)$ is torsion-free [9, (1.3)]. So, by (6.1), the action of $\ker j^*$ produces an infinite number of finitely generated maximal Cohen–Macaulay $A[[T]]$-modules from the module $C[[T]]$, all lifts of C. This shows that lifts of maximal Cohen–Macaulay modules are far from unique.

The first three sections contain definitions, notation, and known results which we will use. In Section 4 we define a divisor class group for rings which satisfy Serre's condition S_2 but are not necessarily normal, and we verify that it is a group. In Sections 5 and 6 we prove the two main results, and in Section 7 we give an example illustrating the second one.

1. Definitions, notation, and relevant results

We begin by reminding the reader of a few elementary definitions without specific reference. Most terms and concepts involved can be found in the texts by Matsumura [15] and Bourbaki [2]. All rings are assumed to be commutative, local, and Noetherian. For any module M over the local ring $(R, \mathfrak{m})$, the depth of M, denoted by $\operatorname{depth}_R M$, is defined to be $\inf\{i \mid \operatorname{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\}$.

The dual M^* of an R-module M is $\operatorname{Hom}_R(M, R)$, and M^{**} means $(M^*)^*$. There is a canonical map $\sigma : M \to M^{**}$. If $\sigma : M \to M^{**}$ is bijective, M is said to be reflexive. Let S be the multiplicatively closed subset of R of the non-zero divisors on R. If the canonical map $\tau : M \to S^{-1}M$ is injective, then M is said to be torsion-free. If $S^{-1}M = 0$, then M is said to be torsion. If R is Gorenstein in codimension one, then M is torsion-free if and only $\sigma : M \to M^{**}$ is injective. If, in addition, R satisfies Serre's condition S_2, then M is torsion-free if and only if M satisfies S_1, and M is reflexive if and only if M satisfies S_2 (see [7, Lemma 3.4 and Theorem 3.6]).

We collect now some facts which can be easily derived from the results in Section 4 of [1].

Proposition 1.1. *If* M *and* N *are finitely generated* R*-modules such that* R *and* N *satisfy* S_2, *then the module* $\operatorname{Hom}(M, N)$ *satisfies* S_2 *and we have an isomorphism* $\operatorname{Hom}(M, N) \cong \operatorname{Hom}(M^{**}, N)$.

Lemma 1.2 (Auslander and Goldman [1]). *Let M and N be finitely generated modules over a Noetherian ring R and $\phi : M \to N$ be a homomorphism such that $\phi_P : M_P \to N_P$ is an isomorphism for any prime ideal P of height less than or equal to one. If M satisfies S_2 and N satisfies S_1, then ϕ is an isomorphism.*

Finally, we present a few homological facts which will be used in the proof of Theorem 5.3.

Proposition 1.3 (Jensen [12, p. 13]). *A directed system $\{\pi_n : M_{n+1} \to M_n\}_{n\geq 1}$ of surjective maps in a category gives rise to the following exact sequence:*

$$0 \to \varprojlim M_n \xrightarrow{\tau} \prod_{n\geq 1} M_n \xrightarrow{\sigma} \prod_{n\geq 1} M_n \to 0,$$

where $\sigma((m_1, m_2, \ldots)) = (m_1 - \pi_1(m_2), m_2 - \pi_2(m_3), \ldots, m_n - \pi_n(m_{n+1}), \ldots)$ and τ is the natural injection.

A short exact sequence of R-modules $0 \to M_1 \to M_2 \to M_3$ is said to be pure exact if $0 \to M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N$ is exact for any R-module N. It is enough to check the condition for any finitely generated module N (see [16, Example 3.40]). Furthermore, the following fact holds (see [12, Proposition 4.5]).

Proposition 1.4. *Let R be any ring. Suppose $0 \to M_1 \to M_2 \to M_3$ is a pure exact sequence of R-modules and T is a finitely presented R-module. Then application of the functor* $\mathrm{Hom}(T, \_\_)$ *to the sequence gives a short exact sequence*

$$0 \to \mathrm{Hom}_R(T, M_1) \to \mathrm{Hom}_R(T, M_2) \to \mathrm{Hom}_R(T, M_3) \to 0.$$

2. More preliminaries: the divisor class group of a normal domain

Let A be a normal domain, and let K be the field of fractions of A. The divisor class group $\mathrm{Cl}(A)$ may be defined as the group of divisorial fractionary ideals $\mathscr{D}(A)$ modulo the subgroup of principal fractionary ideals $\mathscr{P}(A)$ where the multiplication is given by

$$\mathfrak{a} \cdot \mathfrak{b} = [A : [A : \mathfrak{a}\mathfrak{b}]] \quad \text{for } \mathfrak{a}, \mathfrak{b} \in \mathscr{D}(A),$$

where $[M : N]$ denotes the fractionary ideal consisting of the elements x of K such that $xN \subseteq M$. Equivalently, $\mathrm{Cl}(A)$ may be defined as the group of isomorphism classes of A-modules of rank one with multiplication given by $[\mathfrak{a}] \cdot [\mathfrak{b}] = [(\mathfrak{a} \otimes_A \mathfrak{b})^{**}]$. For a fractionary ideal $\mathfrak{a}$, $[\mathfrak{a}]$ denotes the image of $\mathfrak{a}$ in $\mathrm{Cl}(A)$. An element of $\mathrm{Cl}(A)$ is called a divisor class. The product on $\mathrm{Cl}(A)$ coincides with the following one

$$[\mathfrak{a}] \cdot [\mathfrak{b}] = [\mathrm{Hom}_A(\mathfrak{a}^*, \mathfrak{b})].$$

One may consult [2, Ch. VII] or [8] for further details.

It will be useful for our purposes to think of divisor classes in terms of attached divisors, as well. To define these, we first need one more equivalent definition of the divisor class group.

Let $\mathscr{X}^1(A)$ denote the set of height one prime ideals of A. Let $\mathrm{Div}(A)$ denote the free abelian group $\bigoplus_{\mathfrak{p}\in\mathscr{X}^1(A)}\mathbb{Z}$ on the set $\mathscr{X}^1(A)$. Elements of $\mathrm{Div}(A)$ are called divisors of A and are written in the form $\sum_{\mathfrak{p}\in\mathscr{X}^1(A)} n_\mathfrak{p}\cdot\mathfrak{p}$. There is a group isomorphism $\mathrm{div}:\mathscr{D}(A)\to\mathrm{Div}(A)$ defined by

$$\mathrm{div}(\mathfrak{a}) = \sum_{\mathfrak{p}\in\mathscr{X}^1(A)} v_\mathfrak{p}(\mathfrak{a})\cdot\mathfrak{p},$$

where $v_\mathfrak{p}$ denotes the valuation of the discrete valuation ring $A_\mathfrak{p}$. Letting the image of $P(A)$ be denoted by $\mathrm{Prin}(A)$ we get that $\mathrm{Cl}(A)\cong\mathrm{Div}(A)/\mathrm{Prin}(A)$. The canonical surjection from $\mathrm{Div}(A)$ to $\mathrm{Div}(A)/\mathrm{Prin}(A)$ is denoted by cl.

We use additive notation for the product in $\mathrm{Cl}(A)$ when we view it as the group $\mathrm{Div}(A)/\mathrm{Prin}(A)$.

Now for a finitely generated module M over a normal domain there exists a free submodule F of M such that M/F is torsion. The attached divisor of M, denoted by $[M]_{\mathrm{att}}$, is the element $-\mathrm{cl}(\chi(M/F))$ of $\mathrm{Cl}(A)$, where

$$\chi(T) = \sum_{\mathfrak{p}\in\mathscr{X}^1(A)} \ell_{A_\mathfrak{p}}(T_\mathfrak{p})\cdot\mathfrak{p}\in\mathrm{Div}(A),$$

where ℓ denotes the length function. By Proposition 15 of VII Section 4 of [2], it is independent of the choice of F.

The proposition below gives some useful properties of attached divisors.

Proposition 2.1 (Bourbaki [2, Proposition 16 of VII Section 4]). (a) *If* $0\to M_1\to M_2\to M_3\to 0$ *is an exact sequence of finitely generated modules, then* $[M_2]_{\mathrm{att}} = [M_1]_{\mathrm{att}} + [M_3]_{\mathrm{att}}$.

(b) *If there is a homomorphism from* M_1 *to* M_2 *which is an isomorphism in codimension one, then* $[M_1]_{\mathrm{att}} = [M_2]_{\mathrm{att}}$.

(c) *If* $\mathfrak{a}\neq 0$ *is a fractionary ideal of the fraction field* K *of* A, *then* $[\mathfrak{a}]_{\mathrm{att}} = \mathrm{cl}(\mathrm{div}(\mathfrak{a}^{**}))$.

(d) *If* L *is a free module, then* $[L]_{\mathrm{att}} = 0$.

It has the following corollary which we will use.

Proposition 2.2 (Bourbaki [2, Corollary 2 of Proposition 16 of VII Section 4]). *If a divisorial fractionary ideal* $\mathfrak{a}\neq 0$ *has a finite free resolution, then it is principal, i.e.,* $[\mathfrak{a}] = 0$ *in* $\mathrm{Cl}(A)$.

In view of Proposition 2.1, the following result and definition are useful for working with attached divisors.

Proposition 2.3 (Bourbaki [2, Theorem 6 of VII Section 4.9]). *Let M be a finitely generated R-module. Then there is a free module F and a short exact sequence*

$$0 \to F \to M \to J \to 0$$

such that J is an ideal of R.

Definition. Let $\mathscr{P}$ be a property of a finitely generated module or a sequence of finitely generated modules (e.g. exact, split, vanishing, etc.). Pseudo-$\mathscr{P}$ means that the property $\mathscr{P}$ holds in codimension one.

3. More preliminaries: some results on divisor class groups

In this section we give a few results about divisor class groups. Danilov first defined a map $j^* : \mathrm{Cl}(A[[T]]) \to \mathrm{Cl}(A)$ splitting the usual map $i^* : \mathrm{Cl}(A) \to \mathrm{Cl}(A[[T]])$. Subsequently, Melchiors (Special, Aarhus Univ. 1972) demonstrated the existence of j^* purely algebraically (see also [13, Section 1]). The map is defined as follows.

Proposition 3.1. *Let B be a normal domain and t a non-unit such that B/tB is a normal domain. Then there is a canonical group homomorphism $j^* : \mathrm{Cl}(B) \to \mathrm{Cl}(B/tB)$ given by*

$$j^*([\mathfrak{a}]) = [(\mathfrak{a}/t\mathfrak{a})^{**}] = [(\mathfrak{a} \otimes_B B/tB)^{**}]$$

for any reflexive A-module $\mathfrak{a}$ of rank one.

We will need the following result of Griffith and Weston.

Theorem 3.2 (Griffith and Weston [9, Corollary 1.3]). *Let B be an excellent, local, normal domain which is a $\mathbb{Q}$-algebra, and let $t \in B$ be a non-unit such that B/tB is a normal domain. Then the kernel of $j^* : \mathrm{Cl}(B) \to \mathrm{Cl}(B/tB)$ is torsion-free.*

The following characterization by Danilov will be useful for finding examples of rings which have non-discrete divisor class group.

Theorem 3.3 (Danilov [5, Theorem 2]). *Let A be an excellent, local, normal domain which is a $\mathbb{Q}$-algebra. Then A has discrete divisor class group if and only if A satisfies S_3 and the divisor class group $\mathrm{Cl}((A_P)^{\mathrm{sh}})$ of the strict henselisation of A_P is finite for every prime ideal P of A of height less than or equal to two.*

4. Extension of the definition of divisor class group to cover rings satisfying Serre's condition S_2

To embark on an examination of the divisor class groups of normal domains B and B/tB via the rings B/t^nB for $n \geq 1$, we first generalize the notion of class group to cover these rings. A property of a module M is said to hold in codimension i if for each prime ideal P of height less than or equal to i, the property holds for the R_P-module M_P.

Definition. Let A be a ring which satisfies S_2. The divisor class group of A, denoted by $\mathrm{Cl}(A)$, is the set of isomorphism classes of reflexive A-modules which are locally free of rank one in codimension one. The product is given by $[\mathfrak{a}]\cdot[\mathfrak{b}] = [(\mathfrak{a} \otimes_A \mathfrak{b})^{**}]$ for $[\mathfrak{a}], [\mathfrak{b}] \in \mathrm{Cl}(A)$.

As noted in the beginning of Section 2, when A is a normal domain, this definition gives the usual divisor class group (A satisfies R_1); so, there is no contradiction in the notation $\mathrm{Cl}(A)$.

Proposition 4.1. *Let A be a ring which satisfies S_2. Then $\mathrm{Cl}(A)$ as defined above is an abelian group with identity $[A]$ and inverse $[\mathfrak{a}]^{-1} = [\mathfrak{a}^*]$, for $[\mathfrak{a}] \in \mathrm{Cl}(A)$. Also, the assignment $[\mathfrak{a}]\cdot[\mathfrak{b}] = [\mathrm{Hom}_A(\mathfrak{a}^*, \mathfrak{b})]$ gives the same product.*

Proof. First we note that the product is well-defined: $(\mathfrak{a} \otimes \mathfrak{b})^{**}$, the representative for the product of $[\mathfrak{a}]$ and $[\mathfrak{b}]$, is indeed reflexive and locally free of rank one in codimension one.

The same idea is used throughout. To check a group identity, we first construct a natural map between the module representatives of the two sides. Then, using the fact that representatives of elements of $\mathrm{Cl}(A)$ are locally free in codimension one, we show that the map is an isomorphism in codimension one. To finish, we apply Lemma 1.2. □

5. Recovering divisor classes

With our new definition of class group, we can now consider the groups $\mathrm{Cl}(B/t^nB)$ for $n \geq 1$, as long as the rings B/t^nB satisfy the S_2 property. This happens, for example, in the following case: if t is a nonzero-divisor on B such that B/tB satisfies S_2, then B/t^nB satisfies S_2 for all $n \geq 1$.

In Section 3 we saw that if B is a normal domain and t is a non-unit such that B/tB is a normal domain, then the canonical group homomorphism $j^* : \mathrm{Cl}(B) \to \mathrm{Cl}(B/tB)$ is given by

$$j^*([\mathfrak{a}]) = [(\mathfrak{a}/t\mathfrak{a})^{**}] = [(\mathfrak{a} \otimes_B B/tB)^{**}].$$

We define the following additional group homomorphisms.

Lemma 5.1. *Let B be a normal domain and $t \in B$ a nonunit such that B/tB satisfies R_1 and B/t^nB satisfies S_2 for all $n \geq 1$. Then for each $n \geq 1$ there is a group homomorphism*

$$j_n^* : \mathrm{Cl}(B) \to \mathrm{Cl}(B/t^nB)$$

given by $j_n^*([\mathfrak{a}]) = [(\mathfrak{a}/t^n\mathfrak{a})^{**}] = [(\mathfrak{a} \otimes_B B/t^nB)^{**}]$ where the duals are with respect to the ring B/t^nB. Also, j_1^* is exactly j^*.

Proof. We show first that for every height two prime ideal $\tilde{P}$ of B which contains t, $B_{\tilde{P}}$ is regular. Let $\pi : B \twoheadrightarrow B/t^nB$ be the usual surjection. If $\tilde{P}$ is a height two prime ideal of B which contains t, then $P = \pi(\tilde{P})$ is a height one prime ideal of B/t^nB. Since $B_{\tilde{P}}/tB_{\tilde{P}} = (B/tB)_P$ is a regular local ring and t is a non-zero divisor on B, the ring $B_{\tilde{P}}$ is regular.

Therefore, any finitely generated reflexive B-module $\mathfrak{a}$ is locally free at prime ideals of height at most two which contain t, and thus $(\mathfrak{a} \otimes_B B/t^nB)^{**}$ is locally free in codimension one over B/t^nB of the same rank as $\mathfrak{a}$. So, j_n^* is well-defined.

To see that the group structure is preserved by j_n^*, we consider for any $[\mathfrak{a}], [\mathfrak{b}] \in \mathrm{Cl}(B)$ the natural map

$$\theta : ((\mathfrak{a} \otimes_B \mathfrak{b})^{**} \otimes_B B/t^nB)^{**} \to ((\mathfrak{a} \otimes_B B/t^nB)^{**} \otimes_B (\mathfrak{b} \otimes_B B/t^nB)^{**})^{**}$$

which is induced by the maps $\mathfrak{a} \to (\mathfrak{a} \otimes_B B/t^nB)^{**}$ and $\mathfrak{b} \to (\mathfrak{b} \otimes_B B/t^nB)^{**}$. Since $\mathfrak{a}$ and $\mathfrak{b}$ are locally free at height two primes of B which contain t, θ is an isomorphism in codimension one over B/tB. Therefore, by Lemma 1.2, θ is an isomorphism. $\square$

There are also maps between the divisor class groups $\mathrm{Cl}(B/t^nB)$ as follows.

Lemma 5.2. *Let B and t be as in the previous lemma. Then for each $n \geq 1$ there is a group homomorphism*

$$\psi_{n+1} : \mathrm{Cl}(B/t^{n+1}B) \to \mathrm{Cl}(B/t^nB)$$

given by $\psi_{n+1}([\mathfrak{a}]) = [(\mathfrak{a}/t^n\mathfrak{a})^{**}] = [(\mathfrak{a} \otimes_B B/t^nB)^{**}]$, where the duals are with respect to the ring B/t^nB. Furthermore, the maps ψ_{n+1} are compatible with the maps j_n^*.

Proof. The canonical map $\mathrm{Spec}(B/t^nB) \to \mathrm{Spec}(B/t^{n+1}B)$ is a height-preserving homeomorphism. So, if $\mathfrak{a}$ is locally free in codimension one over $B/t^{n+1}B$, then $\mathfrak{a} \otimes_B B/t^nB$ is locally free of the same rank in codimension one over B/t^nB. By an argument similar to the one in the proof of the previous lemma, Lemma 1.2 implies that the maps ψ_{n+1} preserve the group structure and that $\psi_{n+1} \circ j_{n+1}^* = j_n^*$ for all $n \geq 1$.

The main result is the following one.

Theorem 5.3. *Let B be a local normal domain and $t \in B$ be a nonzero divisor such that B is t-adically complete and B/tB is normal. Then the maps j_n^* defined above* *induce an injection*

$$\tilde{j}:\mathrm{Cl}(B) \hookrightarrow \varprojlim \mathrm{Cl}(B/t^nB).$$

Proof. Since $\psi_{n+1} \circ j_{n+1}^* = j_n^*$ for all $n \geq 1$, the image of the map

$$\tilde{j}:\mathrm{Cl}(B) \to \prod_{n\geq 1} \mathrm{Cl}(B/t^nB)$$

is actually in the canonical copy of the inverse limit of the groups $\mathrm{Cl}(B/t^nB)$ in the product $\prod_{n\geq 1} \mathrm{Cl}(B/t^nB)$.

We first discuss a short exact sequence which we will use. Let M be a B-module and $\pi_n : M/t^{n+1}M \twoheadrightarrow M/t^nM$ be the canonical surjections for $n \geq 1$. By Proposition 1.3, these maps give rise to the exact sequence

$$0 \to \varprojlim M/t^nM \xrightarrow{\tau} \prod_{n\geq 1} M/t^nM \xrightarrow{\sigma} \prod_{n\geq 1} M/t^nM \to 0,$$

where $\sigma((m_1, m_2, \ldots)) = (m_1 - \pi_1(m_2), m_2 - \pi_2(m_3), \ldots, m_n - \pi_n(m_{n+1}), \ldots)$ and τ is the natural injection.

If M is finitely generated, then M is t-adically complete, and the sequence becomes

$$0 \to M \xrightarrow{\tau} \prod_{n\geq 1} M/t^nM \xrightarrow{\sigma} \prod_{n\geq 1} M/t^nM \to 0, \qquad (*)$$

where τ is the obvious map. Consider sequence $(*)$ for $M = B$:

$$0 \to B \xrightarrow{\tau} \prod_{n\geq 1} B/t^nB \xrightarrow{\sigma} \prod_{n\geq 1} B/t^nB \to 0. \qquad (**)$$

Since tensoring with finitely presented modules commutes with products, sequence $(**)$ tensored with any finitely generated module M over the Noetherian ring R is just the exact sequence $(*)$. Hence $(**)$ is a pure exact sequence.

Now suppose that $[\mathfrak{a}] \in \mathrm{Cl}(B)$ is in the kernel of $\tilde{j}$. Let B_n denote B/t^nB. Then, for all $n \geq 1$, $[(\mathfrak{a} \otimes_B B_n)^{**}]$ is equal to the identity $[B_n]$ of $\mathrm{Cl}(B_n)$, and so its inverse $[(\mathfrak{a} \otimes_B B_n)^*]$ is equal to the identity $[B_n]$ as well. So, we have the isomorphisms

$$B_n \cong (\mathfrak{a} \otimes B_n)^* = \mathrm{Hom}_{B_n}(\mathfrak{a} \otimes B_n, B_n) \cong \mathrm{Hom}_B(\mathfrak{a}, B_n),$$

for all $n \geq 1$. By Proposition 1.4, an application of $\mathrm{Hom}_B(\mathfrak{a}, \_)$ to the pure exact sequence $(**)$ gives an exact sequence

$$0 \to \mathrm{Hom}(\mathfrak{a}, B) \to \prod_{n\geq 1} \mathrm{Hom}(\mathfrak{a}, B_n) \to \prod_{n\geq 1} \mathrm{Hom}(\mathfrak{a}, B_n) \to 0. \qquad (***)$$

Since $\mathrm{Hom}(\mathfrak{a}, B_n)$ is isomorphic to $B_n = B/t^nB$ and t^n is a non-zero divisor on B, each factor of the product has projective dimension one over B. Since the flat dimension of the factors is bounded by one and since a product of flat modules is flat, the product $\prod_{n\geq 1} \mathrm{Hom}(\mathfrak{a}, B_n)$ has finite flat dimension. Sequence $(***)$ then implies that

$\mathfrak{a}^* = \operatorname{Hom}(\mathfrak{a}, B)$ has finite flat dimension as well. Thus, since every finitely generated R-module of finite flat dimension over a Noetherian ring has finite projective dimension (see [16, 3.58]) $\mathfrak{a}^*$ has finite projective dimension.

Since B is local, Proposition 2.2 implies that $[\mathfrak{a}^*] = 0$. So, indeed, $[\mathfrak{a}] = [\mathfrak{a}^*]^{-1} = 0$, and $\tilde{j}$ is injective. □

Theorem 5.3 translates to the following result in the classical case of $B = A[[T]]$, $t = T$, and $j_1^* = j^* : \mathrm{Cl}(A[[T]]) \to \mathrm{Cl}(A)$.

Corollary 5.4. *For any normal domain A,*

$$\tilde{j} : \mathrm{Cl}(A[[T]]) \hookrightarrow \varprojlim \mathrm{Cl}(A[[T]]/T^n A[[T]])$$

is an injection.

Remark. This is slightly surprising. Danilov showed that, for a large class of rings A, if $\mathrm{Cl}(A[[T]])$ differs from $\mathrm{Cl}(A)$, it differs by an infinite dimensional vector space over $\mathbb{Q}$. So, in those cases $\ker j^* = \ker j_1^*$ is quite large indeed [3]. It is an interesting question whether the map in Theorem 5.3 is a surjection.

6. Lifting maximal Cohen–Macaulay modules

In this section we consider lifts of modules of a very special kind to a deformation of a ring.

Definition. A module M over a local ring $(R, \mathfrak{m})$ is called a Cohen–Macaulay module if $\operatorname{depth}_R M = \dim_R M$. M is called a maximal Cohen–Macaulay module if $\mathfrak{m}M \neq M$ and $\operatorname{depth}_R M = \dim R$.

These modules have been much studied. Hochster constructed maximal Cohen–Macaulay modules over equicharacteristic local rings [11]. On the other hand, it is still unknown whether finitely generated maximal Cohen–Macaulay modules exist in general, except for some specific cases.

We prove the following theorem in order to create, for certain rings B, new finitely generated maximal Cohen–Macaulay B-modules from old ones. Let $\mathfrak{M}_B$ be the set of isomorphism classes of finitely generated maximal Cohen–Macaulay B-modules.

Theorem 6.1. *Suppose B and B/tB are normal domains such that B/tB satisfies R_2. Then the kernel of the map $j^* : \mathrm{Cl}(B) \to \mathrm{Cl}(B/tB)$ acts on $\mathfrak{M}_B$ in the following way: if $[D] \in \mathfrak{M}_B$ and $[\mathfrak{a}] \in \ker j^*$, let*

$$[\mathfrak{a}] \cdot [D] = [\operatorname{Hom}_B(\mathfrak{a}, D)].$$

This action has the following properties:

(a) *If* $[D],[E] \in \mathfrak{M}_B$ *are in the same orbit, then* $D/tD \cong E/tE$.

(b) *Let* $[D] \in \mathfrak{M}_B$ *and* $n = \operatorname{rank}_B(D)$. *If* $[\mathfrak{a}],[\mathfrak{b}] \in \ker j^*$ *are such that* $n[\mathfrak{a}] \neq n[\mathfrak{b}]$, *then* $[\mathfrak{a}] \cdot [D] \neq [\mathfrak{b}] \cdot [D]$.

Proof. Let D be a finitely generated maximal Cohen–Macaulay B-module, and let $[\mathfrak{a}]$ be an element of the kernel of j^*. Let $W_\mathfrak{a}$ denote $\operatorname{Hom}_B(\mathfrak{a}, D)$. Let us first show that $W_\mathfrak{a}$ is a maximal Cohen–Macaulay B-module such that $W_\mathfrak{a}/tW_\mathfrak{a} \cong D/tD$. This involves an argument similar to the one used to prove the main result in [13]. To simplify the notation, let $\overline{B}$ denote B/tB, let $\overline{D}$ denote D/tD, and let $\bar{\mathfrak{a}}$ denote $\mathfrak{a}/t\mathfrak{a}$.

Consider the long exact sequence:

$$\begin{aligned} 0 &\to \operatorname{Hom}_B(\mathfrak{a}, D) \xrightarrow{t} \operatorname{Hom}_B(\mathfrak{a}, D) \to \operatorname{Hom}_B(\mathfrak{a}, \overline{D}) \\ &\to \operatorname{Ext}^1_B(\mathfrak{a}, D) \xrightarrow{t} \operatorname{Ext}^1_B(\mathfrak{a}, D) \to \operatorname{Ext}^1_B(\mathfrak{a}, \overline{D}) \to \cdots \end{aligned}$$

obtained from the short exact sequence $0 \to D \xrightarrow{t} D \to \overline{D} \to 0$. Now, $\operatorname{Hom}_B(\mathfrak{a}, \overline{D}) \cong \operatorname{Hom}_{\overline{B}}(\bar{\mathfrak{a}}, \overline{D}) \cong \operatorname{Hom}_{\overline{B}}(\bar{\mathfrak{a}}^{**}, \overline{D})$, where the second isomorphism holds by Proposition 1.1. But since $[\mathfrak{a}]$ is in $\ker j^*$, we have that $\bar{\mathfrak{a}}^{**} \cong \overline{B}$ and thus that $\operatorname{Hom}_{\overline{B}}(\bar{\mathfrak{a}}^{**}, \overline{D}) \cong \overline{D}$.

The goal is to show that $\operatorname{Ext}^1_B(\mathfrak{a}, \overline{D})$ is zero, for then, by our long exact sequence, $\operatorname{Ext}^1_B(\mathfrak{a}, D)$ would be zero by Nakayama's lemma. Then the sequence

$$0 \to \operatorname{Hom}_B(\mathfrak{a}, D) \xrightarrow{t} \operatorname{Hom}_B(\mathfrak{a}, D) \to \overline{D} > 0$$

would be exact. So, since the module $\overline{D}$ is a finitely generated maximal Cohen–Macaulay B/tB-module, $\operatorname{Hom}_B(\mathfrak{a}, D)$ would have depth equal to $1 + \operatorname{depth}(\overline{D}) = 1 + \dim(\overline{B}) = \dim(B)$. That is, $W_\mathfrak{a} = \operatorname{Hom}_B(\mathfrak{a}, D)$ would be a finitely generated maximal Cohen–Macaulay B-module such that $W_\mathfrak{a}/tW_\mathfrak{a} \cong D/tD$.

Since t is a non-zero divisor on $\mathfrak{a}$ and on B, it follows that, for any projective resolution $P_\bullet$ of $\mathfrak{a}$ over B, $P_\bullet \otimes_B \overline{B}$ is a projective resolution of $\bar{\mathfrak{a}}$ over $\overline{B}$. So, since t annihilates $\overline{D}$, we have the equality $\operatorname{Ext}^1_B(\mathfrak{a}, \overline{D}) = \operatorname{Ext}^1_{\overline{B}}(\bar{\mathfrak{a}}, \overline{D})$.

Since $\mathfrak{a}$ satisfies S_2, $\bar{\mathfrak{a}}$ is a torsion-free $\overline{B}$-module. So, there is an exact sequence of $\overline{B}$-modules

$$0 \to \bar{\mathfrak{a}} \xrightarrow{i} \bar{\mathfrak{a}}^{**} \to \operatorname{coker} \to 0,$$

where $i : \bar{\mathfrak{a}} \to \bar{\mathfrak{a}}^{**}$ is the canonical map of a module to its double dual. Application of $\operatorname{Hom}(\_\_, \overline{D})$ gives the exact sequence

$$\operatorname{Ext}^1_{\overline{B}}(\bar{\mathfrak{a}}^{**}, \overline{D}) \to \operatorname{Ext}^1_{\overline{B}}(\bar{\mathfrak{a}}, \overline{D}) \to \operatorname{Ext}^2_{\overline{B}}(\operatorname{coker}, \overline{D}).$$

Since $[\mathfrak{a}]$ is in $\ker j^*$, $\bar{\mathfrak{a}}^{**} \cong \overline{B}$ and so $\operatorname{Ext}^1_{\overline{B}}(\bar{\mathfrak{a}}^{**}, \overline{D}) = 0$. Hence, in order to show that $\operatorname{Ext}^1_B(\mathfrak{a}, \overline{D}) = 0$, it is enough to show that $\operatorname{Ext}^2_{\overline{B}}(\operatorname{coker}, \overline{D}) = 0$.

This follows from a simple grade argument. First we claim that the support $\operatorname{Supp}(\operatorname{coker})$ lies in codimension three and higher. By hypothesis, for any prime ideal P of $\overline{B}$ such that $\operatorname{ht}(P) \leq 2$, then $\overline{B}_P$ is a regular local ring. Let $\pi : B \twoheadrightarrow \overline{B}$ be the canonical surjection, and let $\tilde{P} = \pi^{-1}(P)$. Now $B_{\tilde{P}}$ is a regular local ring, as well, since its quotient $B_{\tilde{P}}/tB_{\tilde{P}} = \overline{B}_P$ is regular. As such, it has unique factorization, that is,

$\mathrm{Cl}(B_{\bar{P}}) = 0$. Since $\mathfrak{a}_{\bar{P}}$ is reflexive, $[\mathfrak{a}_{\bar{P}}]$ is an element of this trivial class group. So, $\mathfrak{a}_{\bar{P}}$ is free over $B_{\bar{P}}$, and thus $(\bar{\mathfrak{a}})_P = \mathfrak{a}_{\bar{P}} \otimes_B \overline{B}$ is free over $\overline{B}_P = B_{\bar{P}} \otimes_B \overline{B}$. So, the map i is an isomorphism in codimension two over $\overline{B}$, and ht(ann(coker)) is thus at least three. Since $\overline{D}$ is a finitely generated maximal Cohen–Macaulay module, there is a $\overline{D}$-sequence of length at least three in ann(coker). This implies that $\mathrm{Ext}^i_{\overline{B}}(\mathrm{coker}, \overline{D}) = 0$ for $0 \le i \le 2$, as desired.

To see that this is a group action, we note the following isomorphisms:

$$\mathrm{Hom}(A, D) \cong D,$$

$$\mathrm{Hom}(\mathfrak{a}, \mathrm{Hom}(\mathfrak{b}, D)) \cong \mathrm{Hom}(\mathfrak{a} \otimes \mathfrak{b}, D) \cong \mathrm{Hom}((\mathfrak{a} \otimes \mathfrak{b})^{**}, D)$$

for any $[\mathfrak{a}], [\mathfrak{b}] \in \ker j^*$. The second row is the adjoint isomorphism followed by the conclusion of Proposition 1.1. So, indeed,

$$[A] \cdot [D] = [D],$$

$$[\mathfrak{a}] \cdot ([\mathfrak{b}] \cdot [D]) = ([\mathfrak{a}] \cdot [\mathfrak{b}]) \cdot [D].$$

For property (b) of the action of $\ker j^*$, we prove Lemma 6.3 below, which gives a formula for the attached divisor of $\mathrm{Hom}_B(\mathfrak{a}, D)$ for any $[\mathfrak{a}] \in \mathrm{Cl}(B)$. Then for any elements $[\mathfrak{a}]$ and $[\mathfrak{b}]$ in $\mathrm{Cl}(B)$ such that $n[\mathfrak{a}] \neq n[\mathfrak{b}]$, the formula implies that

$$[\mathrm{Hom}_B(\mathfrak{a}, D)]_{\mathrm{att}} = -n\ [\mathfrak{a}]_{\mathrm{att}} + [D]_{\mathrm{att}} \neq -n\ [\mathfrak{b}]_{\mathrm{att}} + [D]_{\mathrm{att}} = [\mathrm{Hom}_B(\mathfrak{b}, D)]_{\mathrm{att}}.$$

So, $\mathrm{Hom}_B(\mathfrak{a}, D)$ and $\mathrm{Hom}_B(\mathfrak{b}, D)$ are indeed non-isomorphic, and so $[\mathfrak{a}] \cdot [D] \neq [\mathfrak{b}] \cdot [D]$. □

In order to prove Lemma 6.3 we need first the following lemma.

Lemma 6.2. *If $\mathscr{E} : 0 \to M \to N \to Q \to 0$ is a pseudo-exact sequence of finitely generated modules over a normal domain R, then $[N]_{\mathrm{att}} = [M]_{\mathrm{att}} + [Q]_{\mathrm{att}}$. Let $\mathscr{F}$ be an additive functor from the category of finitely generated R-modules to itself which commutes with localization. If $\mathscr{E}$ is pseudo-split exact, then $\mathscr{F}(\mathscr{E})$ is pseudo-split exact, and so $[\mathscr{F}(N)]_{\mathrm{att}} = [\mathscr{F}(M)]_{\mathrm{att}} + [\mathscr{F}(Q)]_{\mathrm{att}}$.*

Proof. The first statement is essentially Proposition 2.1. The second is not explicitly stated in Bourbaki, so, since we will need it, we give a quick proof of it: suppose $\mathscr{E} : 0 \to M \to N \to Q \to 0$ is pseudo-split exact. Consider the sequence $\mathscr{F}(\mathscr{E}) : 0 \to \mathscr{F}(M) \to \mathscr{F}(N) \to \mathscr{F}(Q) \to 0$. Let $P \in \mathscr{X}^1(R)$. Since $\mathscr{F}$ commutes with localization, the complex $(\mathscr{F}(\mathscr{E}))_P$ is the same as $\mathscr{F}(\mathscr{E}_P)$. Now, by hypothesis $\mathscr{E}_P$ is split exact. Therefore, since $\mathscr{F}$ is an additive functor, $\mathscr{F}(\mathscr{E}_P)$ is split exact as well. Thus, $\mathscr{F}(\mathscr{E})$ is pseudo-split exact. □

Lemma 6.3. *If I is an ideal of a normal domain R and M is a finitely generated torsion-free R-module of rank n, then*

$$[\mathrm{Hom}_R(I, M)]_{\mathrm{att}} = -n[I]_{\mathrm{att}} + [M]_{\mathrm{att}}.$$

Proof. By Proposition 2.3, there is a short exact sequence

$$0 \to F \to M \to J \to 0$$

such that F is a free module of rank $n-1$ and J is an ideal of R. This sequence is pseudo-split exact, as ideals of R are free in codimension one since they are torsion-free and since R satisfies R_1. By Lemma 6.2 applied twice to the functor $\mathrm{Hom}_R(\_\_, R)$, the sequence

$$0 \to F^{**} \to M^{**} \to J^{**} \to 0$$

is pseudo-split exact. So, as $M^{**} \cong M$ and $F^{**} \cong F$, we may assume J is reflexive when calculating $[\mathrm{Hom}_R(I,M)]_{\mathrm{att}}$. Again by Lemma 6.2, applied to the functor $\mathrm{Hom}_R(I, \_\_)$ this time,

$$\begin{aligned}[\mathrm{Hom}(I,M)]_{\mathrm{att}} &= [\mathrm{Hom}(I,F)]_{\mathrm{att}} + [\mathrm{Hom}(I,J)]_{\mathrm{att}} \\ &= (n-1)\ [I^*]_{\mathrm{att}} + [\mathrm{Hom}(I,J)]_{\mathrm{att}}.\end{aligned}$$

Since $\mathrm{Hom}(I,J)$ satisfies S_2 by Proposition 1.1, we have a canonical isomorphism $\mathrm{Hom}(I,J) \cong (I^* \otimes J)^{**}$. Also, the right-hand side is isomorphic to an ideal of R since it is reflexive and of rank one. So, by part (c) of Proposition 2.1,

$$\begin{aligned}[\mathrm{Hom}(I,J)]_{\mathrm{att}} &= [(I^* \otimes J)^{**}]_{\mathrm{att}} \\ &= \mathrm{cl}(\mathrm{div}(I^* \otimes J)^{**}) \\ &= \mathrm{cl}(\mathrm{div}(I^*)) + \mathrm{cl}(\mathrm{div}(J)) \\ &= [I^*]_{\mathrm{att}} + [J]_{\mathrm{att}}.\end{aligned}$$

Substituting this calculation into the previous one and realizing that $[M]_{\mathrm{att}} = [J]_{\mathrm{att}}$ and $[I^*]_{\mathrm{att}} = -[I]_{\mathrm{att}}$, we get the desired conclusion. □

In the case $B = A[[T]]$ and $t = T$, Theorem 6.1 admits the following corollary.

Corollary 6.4. *Suppose A is a normal domain which satisfies R_2, and C is a finitely generated maximal Cohen–Macaulay A-module. Let $n = \mathrm{rank}_A(C)$. Then for every $[\mathfrak{a}]$ in the kernel of $j^* : \mathrm{Cl}(A[[T]]) \to \mathrm{Cl}(A)$,*

(i) *$W_{\mathfrak{a}} = \mathrm{Hom}_{A[[T]]}(\mathfrak{a}, C[[T]])$ is a finitely generated maximal Cohen–Macaulay $A[[T]]$-module.*

(ii) *$W_{\mathfrak{a}}/tW_{\mathfrak{a}} \cong C$.*

Furthermore, if $[\mathfrak{a}], [\mathfrak{b}] \in \ker j^*$ are such that $n[\mathfrak{a}] \neq n[\mathfrak{b}]$, then $W_{\mathfrak{a}} \ncong W_{\mathfrak{b}}$.

Since by Theorem 3.2 the kernel of $j^* : \mathrm{Cl}(B) \to \mathrm{Cl}(B/tB)$ is torsion-free whenever B and B/tB are excellent normal domains of equicharacteristic zero, the corollary below follows immediately from Theorem 6.1.

Corollary 6.5. *Suppose B is an excellent normal domain of equicharacteristic zero and $t \in B$ is a non-unit such that B/tB is a normal domain which satisfies R_2. If $\ker j^* \neq 0$, then for any finitely generated maximal Cohen–Macaulay B-module D,* *there are infinitely many finitely generated maximal Cohen–Macaulay B-modules W with the property that $W/tW \cong D/tD$.*

Again, in the special case of $B = A[[T]]$, we get a corresponding result. We use the fact that if C is a maximal Cohen–Macaulay A-module, then $C[[T]] = C \otimes_A A[[T]]$ is a maximal Cohen–Macaulay $A[[T]]$-module. By a lift of an A-module C we mean an $A[[T]]$-module D such that $D/tD = C$.

Corollary 6.6. *Suppose A is an excellent normal domain of equicharacteristic zero which satisfies R_2. If $\ker j^* \neq 0$, then any finitely generated maximal Cohen–Macaulay A-module C has infinitely many nonisomorphic lifts to finitely generated maximal Cohen–Macaulay $A[[T]]$-modules.*

7. An example of the lifting

Theorem 3.3 of Danilov, which characterizes excellent normal $\mathbb{Q}$-algebras A with a discrete divisor class group, makes it possible to find examples of rings such that $\ker j^* \neq 0$. In particular, to get an example illustrating Corollary 6.6, we need a local normal domain that satisfies R_2, but not S_3, and has a small maximal Cohen–Macaulay module. We will discuss one such example from a paper by Hochster [10, p. 149].

Let k be an algebraically closed field of characteristic zero. Choose a homogeneous polynomial $f \in k[X_1,X_2,X_3]$ such that $\{X_1,X_2,f\}$ is a system of parameters for $k[X_1,X_2,X_3]$, $X_3^2 \notin (X_1,X_2,f)$, and $A_1 = k[X_1,X_2,X_3]/(f)$ is projectively smooth, i.e., regular away from (X_1,X_2,X_3). For example, let us take $k = \mathbb{C}$ and $f = X_1^3+X_2^3+X_3^3$. Let $A_2 = k[Y_1,Y_2]$. Then the Segre product A_0 of A_1 and A_2 has dimension three and

$$A_0 = k[x_iY_j;\ i = 1,2,3,\ j = 1,2] \hookrightarrow A_1[Y_1,Y_2],$$

where x_i is the image of X_i in A_1. Let A be A_0 localized at the homogeneous maximal ideal $(x_iY_j :\ i = 1,2,3,\ j = 1,2)$. Then A is a local domain with regularity property R_2 (and thus normal), but A does not have the property S_3 as it is not Cohen–Macaulay (see [10]).

To see that A satisfies R_2, we note that A is isomorphic to the ring

$$\left(\frac{k[U,V,W,X,Y,Z]}{(UY-VX, VZ-WY, UZ-WX, U^3+V^3+W^3, X^3+Y^3+Z^3)}\right)_{(U,V,W,X,Y,Z)}.$$

Localization of the three-dimensional ring A at any non-maximal prime ideal P causes one of the generators of the irrelevant maximal ideal to be inverted; by symmetry we may assume it is U. A quick check shows that

$$A\left[\frac{1}{U}\right] = \left(\frac{k[U,V,W,X]}{(U^3+V^3+W^3)}\right)_{(U,V,W,X)}\left[\frac{1}{U}\right].$$

The ring

$$\frac{k[U,V,W]}{(U^3+V^3+W^3)_{(U,V,W)}}$$

is regular away from the ideal (U,V,W), and so $A[1/U]$ and thus also A_P are indeed regular.

Let $Q = Y_1\, A_1[Y_1,Y_2] \cap A_0$. Hochster showed that for large $i > 0$ the symbolic powers of Q are small maximal Cohen–Macaulay A-modules [10, p. 149]. Since A is not S_3, the kernel of j^* is nonzero by Theorem 3.3 and torsion-free by Theorem 3.2. So, Corollary 6.6 implies that there are infinitely many nonisomorphic lifts of that small maximal Cohen–Macaulay A-module to small maximal Cohen–Macaulay modules over $A[[T]]$. This example shows that lifts of maximal Cohen–Macaulay modules are far from unique.

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